Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

MF-33 NYO-9754

GRAVITATIONAL INSTABILITY AND ONE-COMPONENT PLASMA OSCILLATIONS

Richard L. Liboff

November 30, 1962

AEC Research and Development Report

NEW YORK UNIVERSITY

Magneto-Fluid Dynamics Division
Courant Institute of Mathematical Sciences
New York University

TID-4500
18th Edition | Physics

MF-33 | NYO-9754

GRAVITATIONAL INSTABILITY AND ONE-COMPONENT PLASMA OSCILLATIONS

Richard L. Liboff

November 30, 1962

AEC Research and Development Report

Contract No. AT(30-1)-1480

Abstract

A kinetic study is made of a system of neutral particles which interact under gravitational forces. The Jeans instability is recaptured in the limit of long wavelengths where the growth rate assumes a characteristic maximum value. As the wavelength diminishes to a certain critical distance, the instability vanishes. This distance is consistent with thermal-gravitational estimates for the diameter of a star. Plasma oscillations are found in the central region of a one-component charged gas cloud.

Table of Contents

Gravitational Instability and One-Component Plasma Oscillations

I. Introduction and Summary of Results

The equations which govern a "collisionless" one-component gas which is composed of neutral particles are very similar to those which govern a "collisionless" plasma. The single difference resides in the change of sign between the two divergence equations, viz.,

(1) Natural $\nabla \cdot \underset{\sim}{G} = - \alpha\rho$

(2) Charged $\nabla \cdot \underset{\sim}{E} = + \beta q$.

The gravitational force field is $\underset{\sim}{G}$, and $\underset{\sim}{E}$ is the electric field. The mass density is ρ, q is charge density, and α and β are positive constants.

In this paper the consequences of this single difference are examined. Although the plasma problem has been extensively studied in the past score of years (Landau 1946, Van Kampen 1957), interest in the neutral gas problem has only recently been revived (Lynden-Bell 1962, R. Simon 1962), having first been examined in the macroscopic limit by Jeans in 1902. The present work follows quite closely that of Simon.

The first observation is that, while the roots of the Landau problem are obtained only by affecting an analytic continuation, the relevant solutions of the gravity problem are strictly normal modes. Secondly, the neutral gas modes exhibit pure instability

as opposed to the fact that all solutions of the plasma problem are damped. In addition, all of the gravity modes which propagate are damped.

The growing fluctuations are examined in detail. The maximum growth rate of these instabilities is found to be $\omega_0 = \sqrt{\alpha m n_0}$, where m is the mass of the elements and n_0 the equilibrium number density. This maximum growth rate is associated with very long wavelength disturbances -- which is in agreement with Jeans' (1902) conclusions. The growth rate decreases to zero as the wavelength of the disturbance approaches the distance $d = \sqrt{C^2/\omega_0^2}$ (Simon 1962). The mean thermal velocity of the equilibrium distribution is C. The distance d is the analogue of the Debye (1923) distance for plasmas. It is also the well-known lowest order estimate for the dimensions of a star -- which, in turn, is obtained by equating the random thermal energy of a surface particle to its gravitational potential energy.

Finally, a one-component charged spherical plasma is examined, and it is found that plasma oscillations exist in the central region. Such fluctuations should exhibit themselves in a non-charge-neutral universe (Lyttleton and Bondi 1959, and Hoyle 1960).

II. Analysis

Let us consider a neutral gas which is in a Maxwellian state f_o, which exists in the absence of any inter-particle interaction for times $t < 0$. At $t = 0$, the interaction $\underset{\sim}{G}$ is "turned on". The subsequent distribution is $f_o + g$, where $\underset{\sim}{G}$ and g are first order infinitesimals. Neglecting second order terms yields the following equations for g and $\underset{\sim}{G}$:

$$(3) \qquad \frac{\partial}{\partial t} g + \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} g = - \underset{\sim}{G} \cdot \underset{\sim}{\nabla}_{\underset{\sim}{\xi}} f_o \ .$$

The linearization of (1) gives

$$(4) \qquad \underset{\sim}{\nabla} \cdot \underset{\sim}{G} = - \alpha m n_o - \alpha m \int g \, d^2 \xi \ .$$

The mass of the particles is m, and the distribution function f_o is normalized over velocity ξ to the number density n_o.

If the plane wave forms, $\exp\left\{ -i\omega t + i\underset{\sim}{k} \cdot \underset{\sim}{x} \right\}$, are substituted into the above set, there results

$$(5) \qquad \left\{ \alpha m \int \frac{\underset{\sim}{\nabla}_{\xi} f_o \, d^3 \xi}{\underset{\sim}{k} \cdot \underset{\sim}{\xi} - \omega} + \underset{\sim}{k} \right\} \cdot \underset{\sim}{G} = i \alpha m n_o \ .$$

For longitudinal ($\underset{\sim}{k} \times \underset{\sim}{G} = 0$) fluctuations, the following dispersion relation is obtained from the above:

$$(6) \qquad k^2 = \int_{-\infty}^{\infty} \frac{v \, \phi \, dv}{v - (\omega/k)} \ .$$

This equation is in purely non-dimensional form. The (non-dimensional) wave number k is written for kd, the frequency ω is written for (ω/ω_o), and the velocity v is written for ξ_z/C. The function ϕ is the one-dimensional distribution function,

$$(7) \qquad \phi(v) = (2\pi)^{-1/2} e^{-v^2/2} = (2\pi)^{-3/2} C^{-3} \iint d\xi_x d\xi_y \, e^{-\xi^2/2C^2} .$$

The distance d is the analogue of the Debye distance in plasmas (P. Debye and E. Hückle 1923), while the growth rate ω_o is the analogue of the characteristic frequency of plasmas (L. Tonks and I. Langmuir 1929):

$$(8) \qquad \omega_o^2 = \alpha m n_o ,$$

$$(9) \qquad d^2 = C^2/\omega_o^2 .$$

Equation (6) will have roots only when the function

$$(10) \qquad F(z) \equiv \int_{-\infty}^{\infty} \frac{v \, \phi \, dv}{v-z} , \quad z = \omega/k$$

is real and positive.

It is easily shown (see Appendix) that $\operatorname{Im} F > 0$ in the first quadrant of the z-plane (including the positive real axis), $\operatorname{Im} F < 0$ in the second quadrant (including the negative real axis), and $\operatorname{Im} F = 0$ along the imaginary axis. It follows

that all the unstable roots of (6) (i.e., roots lying in the upper half plane) lie along the imaginary axis, or equivalently for

$$(11) \qquad z = i\beta \ , \qquad \beta = |\beta| \ .$$

For these values of z, F assumes the form,

$$(12) \qquad F(i\beta) = \int_{-\infty}^{\infty} \frac{v^2 \phi \, dv}{v^2 + \beta^2} \ .$$

If k^2 is rewritten as

$$(13) \qquad k^2 = \nu^2/\beta^2 \ ,$$

where ν is the growth rate,

$$(14) \qquad \omega = i\nu$$

(all values are non-dimensional through d and ω_o), then (6) now appears as

$$(15) \qquad (\nu^2/\beta^2) = \int_{-\infty}^{\infty} \frac{v^2 \phi \, dv}{v^2 + \beta^2} \ .$$

For a given frequency ν, this equation determines β which, in turn, through (13), gives k. Both sides of (15) are plotted in Fig. 1.

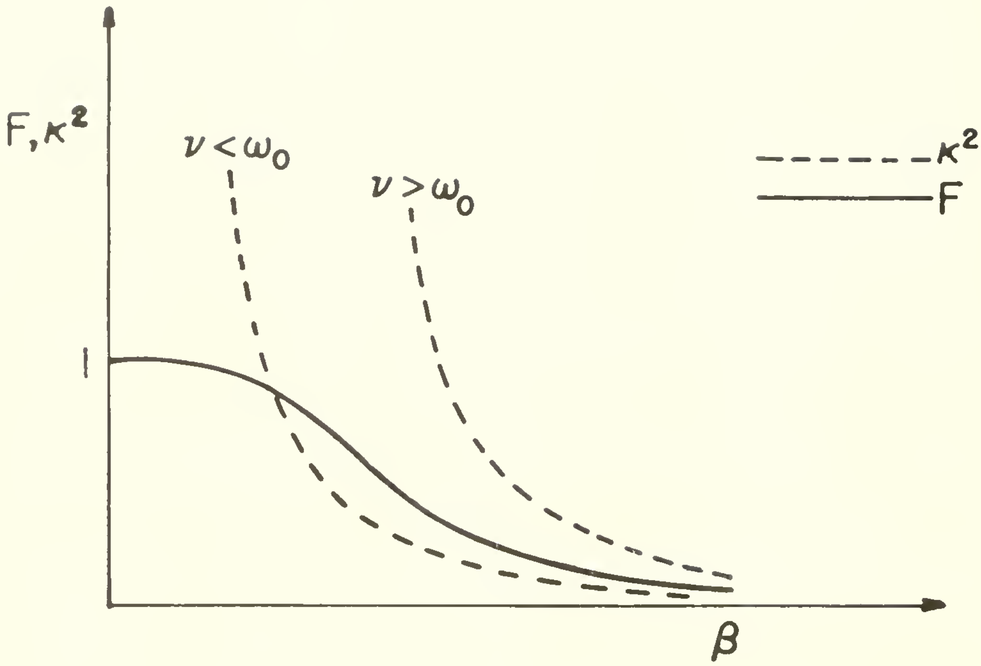

Figure 1

The inequalities which are depicted in Fig. 1 are obtained by examining the asymptotic behavior of F for large β, according to which

(16) $$F \sim \frac{1}{\beta^2}\left(1 - \frac{3}{\beta^2} + \dots\right) ,$$

(13) $$k^2 = \nu^2/\beta^2 .$$

It follows that, for large β, both k^2 and the first derivative of k^2 are larger than the related values of F if

$$\nu^2 > 1 ,$$

which appears as

$$\nu > \omega_o \tag{17}$$

in dimensional form. It follows that there are no solutions for $\nu > \omega_o$, so that ω_o represents the maximum growth rate of these instabilities.

The relevant root for large β is obtained by equating equations (13) and (16). There results (in dimensional form),

$$\nu^2 = \omega_o^2 - 3k^2C^2 \ , \tag{18}$$

which is identical to Jeans' result. These values of ν are close to the largest (i.e., ω_o) and clearly relate to $k \approx 0$, or, equivalently, to $\lambda \approx \infty$, which is again consistent with Jeans' conclusions.

The smallest ν values lie at the opposite extreme, $\beta \sim 0$. The expansion of $F(i\beta)$ about small β appears as

$$\begin{aligned} F &\sim 1 - \beta^2 P: \int_{-\infty}^{\infty} \frac{\phi}{v^2}\, dv + \ldots - \beta\sqrt{\pi/2}\, e^{\beta^2/2} \\ &= 1 - \beta\sqrt{\pi/2} + \ldots \ . \end{aligned} \tag{19}$$

The symbol $P:$ indicates that the principal value is to be taken. Equating the above series to k^2 gives the desired roots,

$$k^2 = \nu^2/\beta^2 = 1 - \beta + \ldots \ . \tag{20}$$

It follows that in the limit $\beta \to 0$, $\nu \to 0$ and* $k \to 1$. In dimensional form,

$$\nu << \omega_o \, , \quad k \to d^{-1} \, . \tag{21}$$

We conclude that, although long wavelength disturbances grow most rapidly, this growth rate diminishes to zero as the distance between condensations approaches the critical distance d. In a homogeneous gas of infinite extent the characteristic time for the development of this final state is ω_o^{-1}.

All of the remaining modes are damped. The exact forms are obtained by continuing F analytically into the lower half z-plane. The only physically relevant fact about these damped modes is that some of them propagate (viz., the roots that lie off the imaginary axis).

In the one-component charged gas problem, the $\underset{\sim}{E}$ field, due to the Coulomb interaction, is very much larger than the Newtonian interaction force encountered above, so that a perturbation technique such as was used in the neutral gas problem would appear inconsistent in the charged gas case. However, as is well known, the Coulomb force becomes vanishingly small in the center of a charged spherical cloud, being of the order (r/R), where R is the radius of the cloud and r the radius of the

---

* That k^2 lies in the range $0 \leq k^2 \leq 1$ is most easily seen by plotting k^2 = constant vs. F (cf. Fig. 1).

said central region. The related equations are identical to the set (2, 3) with $\underset{\sim}{G}$ replaced by $\underset{\sim}{E}$ and α by $-\beta$. As Landau (1946) has shown, all solutions are damped. In the limit of long wavelengths, the damping is exponentially small, and the roots appear as

$$\omega^2 = \omega_P^2 + 3k^2C^2 \,, \tag{22}$$

a propagating oscillatory mode. Although these modes are quite well known, they are usually obtained assuming charge-neutrality in the equilibrium configuration. The classical plasma frequency is ω_P (Tonks and Langmuir 1929).

Acknowledgement

Useful discussions with Professors H. Grad and S. Korf are gratefully acknowledged. The author is particularly indebted to Professor H. Weitzner for rendering critical comments which greatly clarified the nature of these results.

Appendix

The following arguments exhibiting the properties of $F(z)$ are due to Professor H. Weitzner.

First we recall that $F(z)$ is defined as

$$(A.1) \qquad F(z) = \int_{-\infty}^{\infty} \frac{v\ e^{-v^2/2}\ dv}{v-z} .$$

If the integrand is multiplied by $1 = (v+z)/(v+z)$, there results

$$(A.2) \qquad F(z) = \int_{-\infty}^{\infty} \frac{v^2\ e^{-v^2}}{v^2 - z^2} ,$$

where the remaining term is dropped because of oddness of the integrand. Multiplying again by $1 = (v^2 - \bar{z}^2)/(v^2 - \bar{z}^2)$ gives the form

$$(A.3) \qquad F(z) = \int \frac{v^2\ e^{-v^2}(v^2 - \bar{z}^2)}{|v^2 - \bar{z}^2|^2} .$$

It follows that

$$(A.4) \qquad \operatorname{Im} F(z) = - \alpha \operatorname{Im} \bar{z}^2 ,$$

where α is the real positive number,

$$(A.5) \qquad \alpha = \int_{-\infty}^{\infty} \frac{v^2\ e^{-v^2/2}\ dv}{|v^2 - z^2|^2} .$$

Setting $z = re^{i\theta}$, one obtains

(A.6) $$\frac{1}{\alpha r^2} \operatorname{Im} F = \sin 2\theta \ .$$

For values of z along the real axis, F appears as

(A.7) $$F(x) = P: \int_{-\infty}^{\infty} \frac{v e^{-v^2/2}\, dv}{v-x} + 2\pi i x\, e^{-x^2/2} \ .$$

The principal part $P:$ is purely real while the Cauchy contribution is purely imaginary, so that for $\theta = 0, \pi$,

(A.8) $$\operatorname{Im} F = 2\pi x\, e^{-x^2} \ .$$

The value of F along the positive imaginary axis has been calculated in the text. It follows that in the entire upper half z-plane,

(A.9) $$\begin{array}{ll} \operatorname{Im} F > 0 & 0 \leq \theta < \pi/2 \\ \operatorname{Im} F = 0 & \theta = \pi/2 \\ \operatorname{Im} F < 0 & \pi/2 < \theta \leq \pi \ . \end{array}$$

Q.E.D.

References

1. Debye, P. and E. Hückle, Physik. Z. 24, 185 (1923).
2. Hoyle, F., Proc. Roy. Soc. A, 257, 431 (1960).
3. Jeans, J., Phil. Trans. 199, A, 49 (1902).
4. Landau, L., J. Phys. (USSR) 10, 25 (1946).
5. Lynden-Bell, D., Monthly Notices, Roy. Ast. Soc. 124, 279 (1962).
6. Lyttleton, R. A. and H. Bondi, Proc. Roy. Soc. A, 252, 313 (1959).
7. Simon, R., Bull. de l'Acad. Royal de Belgique, Series 5, 47, 7 (1962).
8. Tonks, L. and I. Langmuir, Phys. Rev. 33, 195 (1929).
9. Van Kampen, N. G., Physica XXIII, 641 (1957).